The basics of integration (beta version)

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14 października 2006

1 Derivative

The derivative of a function $F(x)$ with respect to its variable x denotes the rate of change of this function:

$$F'(x) = \frac{d}{dx}F = \frac{dF}{dx} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} \tag{1}$$

Here you see various notations for the derivative.

Note: when the function is a constant function, then $F(x + \Delta x) = F(x)$, and the derivative is zero. On the other hand, when the function grows rapidly, $F(x+\Delta x) > F(x)$ and the derivative is positive. When the function decreases, $F(x + \Delta x) < F(x)$ and the drivative is negative.

We can calculate any possible derivative using this limit formula. For some functions this is already done for us by the mathematicians. The most important derivatives are:

$$(x^n)' = \frac{d(x^n)}{dx} = nx^{n-1} \tag{2}$$

$$(\sin x)' = \frac{d(\sin x)}{dx} = \cos x \tag{3}$$

$$(\cos x)' = \frac{d(\cos x)}{dx} = -\sin x \tag{4}$$

$$(\exp(x))' = \frac{d(\exp(x))}{dx} = \exp(x) \tag{5}$$

$$(\log x)' = \frac{d(\log x)}{dx} = \frac{1}{x} \tag{6}$$

(In this table you can see the two different notations for derivative–one with prime, second with the “d” sybmol).

Differentiation (the process of taking a derivative) is a linear process. This means that the following relations hold (k being a constant):

$$(f(x)+g(x))' = (f(x))' + (g(x))' \tag{7}$$
$$(kf(x))' = k(f(x))' \tag{8}$$

2 Indefinite integral

The indefinite integration is a process opposite to differentiation. It allows to obtain a function, that was differentiated., i.e:

$$(F(x))' = f(x) \leftrightarrow \int f(x)dx = F(x) + c \tag{9}$$

In the process of integration above, we have recovered the function F, that after differentiation gives you f. We have recovered however this function plus some constant c. This is because if we differentiate a constant with respect to x, we obtain zero (see the limit definition of derivative). Thus not only a function F after differentiation gives f, but also $F + c$! This is why we need to include integration constant.

The dx mark in the integration sign is a "dummy variable"–it shows with respect to what variable we preform integration (here with respect to x).

We can write the integral above in another form, to show some relations to the derivative and to the "d" symbols:

$$\int f(x)dx = \int F'(x)dx = \int \frac{dF}{dx}dx = \int dF \tag{10}$$

The dF is the infinitly small increase in the value of function F (recall the relation of dF and ΔF). The integral, being written in a stylish S is not written so accidentally. This S referres to the summation of dF pieces to build in sum the complete value F. This is specially important interpretation when considering definite integrals.

Also note that $\int dF$ is the integral of function 1 where the dummy variable is dF. The integral of the function 1 equals x, as in the table below, but since our dummy variable is dF, not dx, here the integral equals F. You see: the intuitive interpretation of F, being a sum of small pieces dF got explanation in a formal integration process.

Applying the antidifferentiation idea to the table of functions, that we had in the differentiation section, we can construct an integration table, i.e.:

$$\begin{aligned}
\int x^n dx &= \frac{1}{n+1}x^{n+1} + c && (11)\\
\int \cos x dx &= \sin x + c && (12)\\
\int \sin x dx &= -\cos x + c && (13)\\
\int \exp(x) dx &= \exp(x) + c && (14)\\
\int \frac{1}{x} dx &= \log(x) + c && (15)
\end{aligned}$$

The integration is a linear operation, and it underlies the linear laws, similarly as the differentiation:

$$\begin{aligned}
\int f(x) + g(x) dx &= \int f(x)dx + \int g(x)dx && (16)\\
\int kf(x)dx &= k\int f(x) && (17)
\end{aligned}$$

which is helpful in calculations.

3 Definite integration

Definite integration is a process that includes indefinite integration as one of the stages. The definition of a definite integral can be seen as follows:

$$(F(x))' = f(x) \leftrightarrow \int_a^b f(x)dx = F(b) - F(a) \qquad (18)$$

The process of obtaining the function F is exactly the indefinite integration! However, after we obtain this function, we don't simply write it, but we substitute b and a for the arguments in this function, and write the result as the subtraction. The result is a NUMBER this time, not FUNCTION of x, as in the indefinite integration!

The interpretation of definite integral over a function $f(x)$ has a geometrical meaning. the definite integral in range from a to b (which are the integration boundaries) equals to the AREA BELOW the function $f(x)$ in this range. This can be easily understood by recalling the summation interpretation of integration. The $f(x)dx$ term is the area below the curve $f(x)$ at point x in the range $(x, x + dx)$ (the function has not enough space to change in infinitly small dx, so we can calculate this area as the area of a bar). Summing up the areas in subsequent dx intervals, we recover the total area below the curve.